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by
RICHARD E. BARLOW
and
ESTHER SID HUDES

OPERATIONS
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UNDER ALTERNATIVE OPERATING RULES, I

by

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ABSTRACT

This is the first part of a study of stochastic processes generated by failures and repairs of components in a series system. Selected asymptotic measures of system performance are calculated for a two component series system where failure of component 1 shuts off component 2 but not vice versa. Intuitively, it is best to shut off an operating component when the system is down. However, this is not correct if the component failure distribution is IFR (or more generally NBUE) and we wish to maximize the long run average of system uptimes. Part 2 uses the method of supplementary variables to obtain explicit results for important special cases.

ASYMPTOTIC MEASURES OF SYSTEM PERFORMANCE UNDER ALTERNATIVE OPERATING RULES, I

by

Richard E. Barlow and Esther Sid Hudes

1. INTRODUCTION AND SUMMARY

In this paper we study stochastic processes generated by failures and repairs of components in a series system. A series system of k components operates if and only if each of the k components operates. However, depending on the shut-off rule, some components may continue to operate with the system down. For example, failure of the power supply may shut down a computer but not vice versa. Only failed components are repaired or replaced, and repair or replacement takes a random time. Repaired components are assumed to function like new components. Furthermore, components are separately maintained. Failure and repair times are statistically independent.

We are interested in the asymptotic (as time becomes infinite) values of selected measures of system performance. Some of these quantities are:

(i) The limiting system availablity; that is, the limiting probability that the system is functioning;

(ii) The limiting system failure rate;

(iii) The limiting average of system uptimes and downtimes;

(iv) The limiting average number of system failures due to a specified component.

These measures have various uses. For example, (iv) might be used to evaluate the relative importance of different components in the system.

There is a large literature on this subject and we mention only recent related work. Unless otherwise mentioned, we assume that all component failure distributions are continuous so that the probability of simultaneous component failures is zero.

Model A: Components Operate Independently

Suppose nonfailed component i $(i = 1, \ldots, k)$ continues to operate regardless of the number of failed components. This model was treated for arbitrary coherent structures by Ross (1975). Let component i have mean life μ_i and mean repair time ν_i $(i = 1, \ldots, k)$. Let $\tilde{N}_i(t)$ be the number of system failures in $[0,t]$ caused by component i and let $\tilde{N}(t)$ be the number of syst failures in $[0,t]$. Obviously $\tilde{N}(t) = \sum_{i=1}^{k} \tilde{N}_i(t)$.

(i) The limiting series system availability is

$$A = \prod_{i=1}^{k} \left(\frac{1}{1 + \frac{\nu_i}{\mu_i}} \right);$$

(ii) The limiting series system failure rate is

$$\lim_{t \to \infty} \frac{\tilde{N}(t)}{t} = A \sum_{i=1}^{k} \frac{1}{\mu_i};$$

(iii) Let μ (ν) be the limiting average of system uptimes (downtimes). Then

$$\mu = 1 \Big/ \sum_{i=1}^{k} \frac{1}{\mu_i}$$

and

$$\nu = \frac{1 - A}{A \sum_{i=1}^{k} \frac{1}{\mu_i}} = \mu \left[\prod_{i=1}^{k} \left(1 + \frac{\nu_i}{\mu_i}\right) - 1 \right] ;$$

(iv) The limiting average number of system failures due to component i is

$$\lim_{t\to\infty} \frac{\tilde{N}_i(t)}{t} = A/\mu_i \ .$$

The above results hold under very general conditions on the life and repair distributions, i.e., when the convolution $F_i * G_i$ $(i = 1, \ldots, k)$ is nonlattice.

Model B is the other extreme to Model A so far as shut-off rules are concerned.

Model B: Nonfailed Components in Suspended Animation During System Downtime

Barlow and Proschan (1973) obtained the previous measures of performance for series systems for which no component operates while the system is down. Nonfailed components do not age (are in "suspended animation") during system downtime.

(i) The limiting system availability for this model is

$$A = \frac{1}{1 + \sum_{i=1}^{k} \frac{\nu_i}{\mu_i}} ;$$

(ii) The limiting system failure rate is

$$\lim_{t\to\infty} \frac{\tilde{N}(t)}{t} = A \sum_{i=1}^{k} \frac{1}{\mu_i} ;$$

(iii) $\mu = \frac{1}{\sum_{i=1}^{k} \frac{1}{\mu_i}}$, as in Model A, but

$$\nu = \frac{1 - A}{A \sum_{i=1}^{k} \frac{1}{\mu_i}} = \mu \sum_{i=1}^{k} \frac{\nu_i}{\mu_i} ;$$

(iv) The limiting average number of system failures due to component i is

$$\lim_{t\to\infty} \frac{\tilde{N}_i(t)}{t} = A/\mu_i .$$

It is our objective to obtain the previous measures of system performance for series systems whose components are subject to various shut-off rules. For every component pair (i,j) we assume a shut-off rule which determines whether or not j is shut off when i fails and vice versa. By the notation

$$\underset{i}{o} \longrightarrow \underset{j}{o}$$

we mean that when i fails, j is shut off (if j is working) but not vice versa. We assume that j cannot fail and does not age while it is shut off. This assumption is not always realized in practice, since the act of shutting off and turning on a component can sometimes introduce stresses on the component.

By the notation

$$\underset{1}{\circ} \quad \underset{2}{\circ}$$

we mean that components 1 and 2 operate independently of each other. For all shut-off rules, we assume that components are separately maintained; i.e., it is as if each has a separate and independent repairman.

All possible shut-off rules (up to component permutation) are exhibited in Figure 1.1 for $k = 2$ and $k = 3$. Obviously the number of possibilities increases very rapidly with k . Hence, we concentrate on the case $k = 2$ with shut-off rule

$$\underset{1}{\circ} \!\rightarrow\! \underset{2}{\circ}$$

for general failure and repair distributions.

The basic results obtained are that

- Asymptotic measures of system performance defined above depend on the Laplace transforms of probability distributions at specified values of the argument as well as means;
- Sharp bounds on asymptotic measures of system performance can be given in terms of means assuming certain distributions belong to the class of "new better (worse) than used in expectation" and "new better (worse) than used." [Marshall and Proschan (1972).]

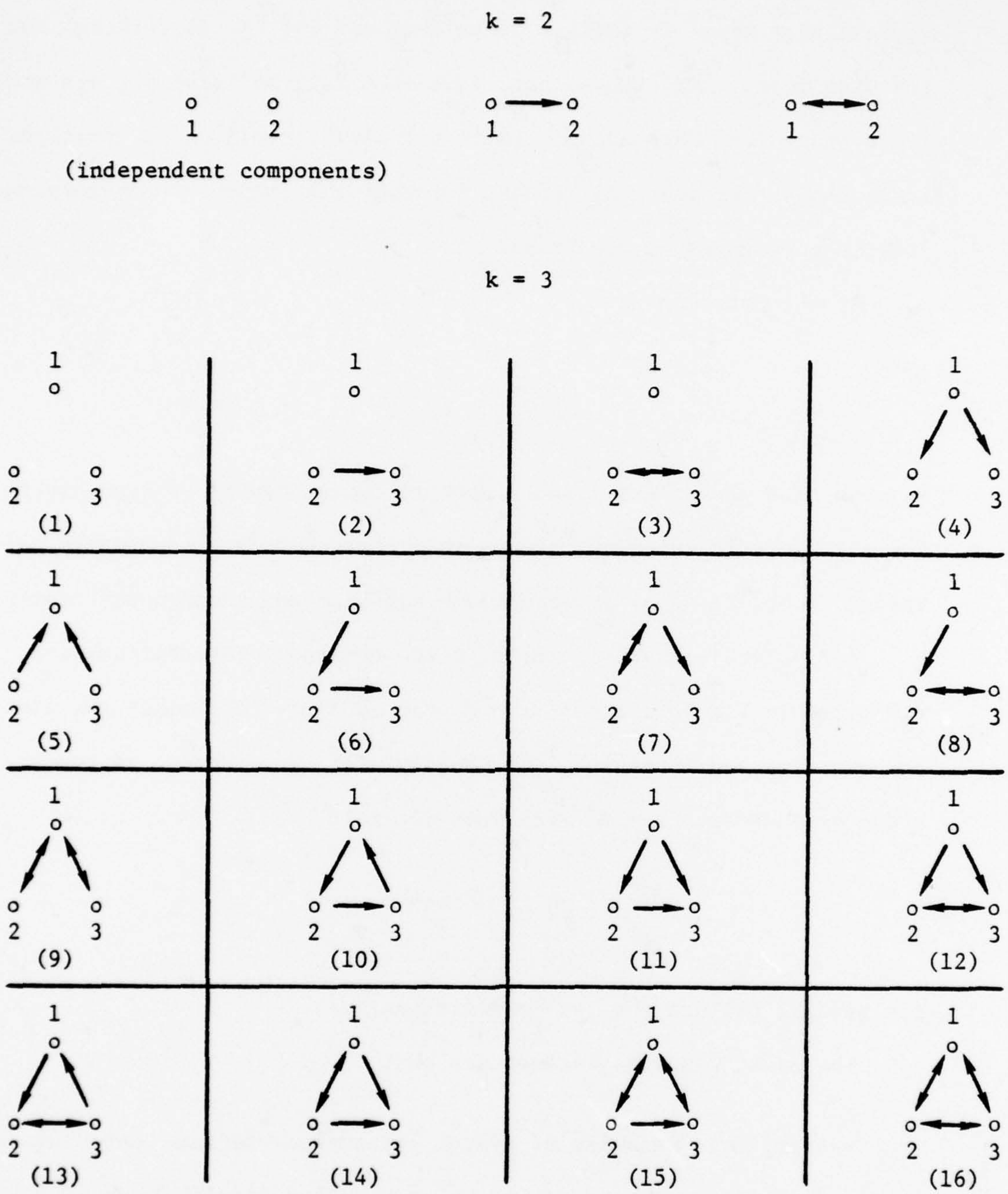

FIGURE 1.1

SHUT-OFF RULES FOR SERIES SYSTEMS

We assume that component i has failure distribution F_i with mean μ_i and repair distribution G_i with mean ν_i , $i = 1,2$. Limiting system availability is difficult to compute for the general case. However, if F_1 and G_1 are exponential, then

$$\text{(i)} \qquad A(F_2,G_2) = \frac{\mu_1}{\mu_1 + \nu_1}\left\{\frac{1}{1 + \frac{\nu_2}{\mu_2}\left[\frac{\mu_1}{\mu_1 + \nu_1} + \frac{\nu_1}{\mu_1 + \nu_1}\,\frac{1}{\nu_2}\,\bar{G}_2^*\left(\frac{1}{\mu_1} + \frac{1}{\nu_1}\right)\right]}\right\}.$$

$A(F_i,G_j)$ is the limiting system availability when all distributions but F_i and G_j are assumed to be exponential. $\bar{G}_2^*$ is the Laplace transforms of $\bar{G}_2 = 1 - G_2$.

Note the interesting fact that $A(F_2,G_2)$ depends on F_2 only through the mean μ_2 , while the dependence on G_2 is much greater--through its Laplace transform at $\frac{1}{\mu_1} + \frac{1}{\nu_1}$ (in addition to its mean, ν_2) .

Assuming only that F_1 is exponential we can show that

(ii) The limiting system failure rate is

$$\lim_{t\to\infty} \frac{\tilde{N}(t)}{t} = A(G_1,F_2,G_2) \cdot \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right);$$

(iii)
$$\mu = \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2}}$$

$$\nu = \frac{1 - A(G_1,F_2,G_2)}{A(G_1,F_2,G_2)\left[\frac{1}{\mu_1} + \frac{1}{\mu_2}\right]};$$

(iv) The limiting number of system failures due to component i is

$$\lim_{t\to\infty} \frac{\tilde{N}_i(t)}{t} = A(G_1,F_2,G_2)/\mu_i \ ;$$

$i = 1,2$.

(Note that the limiting failure rate and μ are similar for Models A, B and this case when $k = 2$. Unfortunately, when F_1 is not exponential, the similarity no longer exists.)

(v) $A_{..} \le A_{\rightarrow} \le A_{\leftrightarrow}$.

These inequalities are intuitively appealing, since the more components we shut off during system downtime, the more component lifetime we save. We conjecture that (v) holds in general, but were able to prove only the left hand inequality. The right side inequality holds for special cases, e.g., for $A(F_1,G_1)$ when either F_1 or G_1 is new worse than used in expectation (NWUE).

(vi) $\nu_{..} \ge \nu_{\rightarrow} \ge \nu_{\leftrightarrow}$.

These inequalities follow from (v), since, when F_1 is exponential, we have equality of μ for the three shut-off rules.

No such clear relationships exist when F_1 is general, since μ is then different from $\dfrac{1}{\frac{1}{\mu_1}+\frac{1}{\mu_2}}$ under the rule that 1 shuts off 2, and can be either larger or smaller.

For example, if F_1 is NBUE (NWUE), G_1 general, and F_2 , G_2 exponential, then

$$\mu_{\rightarrow}(F_1,G_1) \ge (\le) \frac{1}{\frac{1}{\mu_1}+\frac{1}{\mu_2}} \ .$$

NBUE means new better than used in expectation. This means that if it is important to have long operating periods, then when F_1 is NBUE, G_1 general, and F_2 , G_2 exponential, we are better off *not* shutting off component 1 when component 2 fails.

It follows from (i) that if G_2 is IFRA, then

$$A\left(D_{\nu_2}\right) \leq A(F_2,G_2) \leq A(\exp)$$

where $A(\exp)$ is the limiting availability when all distributions are exponential and $A\left(D_{\nu_2}\right)$ is the limiting availability when G_2 is degenerate at ν_2 and all other distributions are exponential. The bounds are sharp.

Since often exact distributions are not known even though means may be specified, the bounds based only on means and NBU (NWU) or NBUE (NWUE) assumptions may be of some practical use.

Analogously to (i), if one assumes only that F_2 and G_2 are exponential, we get

$$A(F_1,G_1) = \frac{\mu_1}{\mu_1 + \nu_1} \times$$

$$\left\{ \frac{1}{1+\frac{\nu_2}{\mu_2}\left[\frac{\bar{F}_1^*\left(\frac{1}{\mu_2}+\frac{1}{\nu_2}\right)+\frac{\mu_2}{\mu_2+\nu_2}\,\bar{G}_1^*\left(\frac{1}{\nu_2}\right)-\left(\frac{1}{\nu_2}+\frac{1}{\mu_1}\,\frac{\mu_2}{\mu_2+\nu_2}\right)\bar{F}_1^*\left(\frac{1}{\mu_2}+\frac{1}{\nu_2}\right)\bar{G}_1^*\left(\frac{1}{\nu_2}\right)}{\bar{F}_1^*\left(\frac{1}{\mu_2}+\frac{1}{\nu_2}\right)+\frac{\mu_2}{\mu_2+\nu_2}\,\bar{G}_1^*\left(\frac{1}{\nu_2}\right)-\left(\frac{1}{\nu_2}-\frac{1}{\mu_1}\,\frac{\mu_2}{\mu_2+\nu_2}\right)\bar{F}_1^*\left(\frac{1}{\mu_2}+\frac{1}{\nu_2}\right)\bar{G}_1^*\left(\frac{1}{\nu_2}\right)}\right]} \right\}$$

and from this we can also obtain

$$A(F_1) = \frac{\mu_1}{\mu_1 + \nu_1}\left\{\frac{1}{1 + \frac{\nu_2}{\mu_2}\left[\dfrac{\frac{\mu_2}{\mu_2+\nu_2} + \left(\frac{1}{\nu_1} - \frac{1}{\mu_1}\frac{\mu_2}{\mu_2+\nu_2}\right)\bar{F}_1^*\left(\frac{1}{\mu_2}+\frac{1}{\nu_2}\right)}{\frac{\mu_2}{\mu_2+\nu_2} + \left(\frac{1}{\nu_1} + \frac{1}{\mu_1}\frac{\nu_2}{\mu_2+\nu_2}\right)\bar{F}_1^*\left(\frac{1}{\mu_2}+\frac{1}{\nu_2}\right)}\right]}\right\}$$

and

$$A(G_1) = \frac{\mu_1}{\mu_1 + \nu_1}\left\{\frac{1}{1 + \frac{\nu_2}{\mu_2}\left[\dfrac{1}{1 + \frac{1}{\mu_1}\bar{G}_1^*\left(\frac{1}{\nu_2}\right)}\right]}\right\} .$$

The expressions for the limiting system failure rate, μ and ν are no longer similar to those in Models A and B, nor are they simple and intuitive.

Here, too, we can bound the availability $A(F_1,G_1)$ if one or two distributions is known to be NBU (NWU).

If F_1 is NBU (NWU) then

$$A\left(D_{\mu_1},G_1\right) \geq A(F_1,G_1) \geq (\leq)\ A(G_1) .$$

If G_1 is NBU (NWU) then

$$A\left(F_1,D_{\nu_1}\right) \geq A(F_1,G_1) \geq (\leq)\ A(F_1) .$$

2. ASYMPTOTIC RESULTS FOR THE TWO UNIT CASE

2.1 Preliminaries

Consider a structure of two components in series. The component in position i operates a random length of time, then fails and is replaced or repaired during a random length of time with distribution G_i and mean ν_i , $0 < \nu_i < \infty$. The operating lifetime of component i has distribution F_i and mean μ_i $(i = 1,2)$. All random variables are assumed independent. Components are separately maintained (i.e., it is as if each had its own repair facility).

We call the component in position i , component i $(i = 1,2)$ even though, due to failure, it may not be the original component in that position. Component 1 can fail at any instant in time that it is not under repair. If component 1 fails when component 2 is working, component 2 is shut off and enters a state of "suspended animation" during which time it cannot fail and does not age. When component 1 is repaired, both components are switched back on. If component 2 is undergoing repair when component 1 fails, component 2 will complete its repair. If component 2 completes its repair before component 1 is fixed, component 2 will remain shut off until 1 is fixed.

Figure 2.1 illustrates these ideas in terms of a sample history.

Let $\{Z(t) ; t \geq 0\}$ be the stochastic process with state space $\{0,1,2,3\}$ where

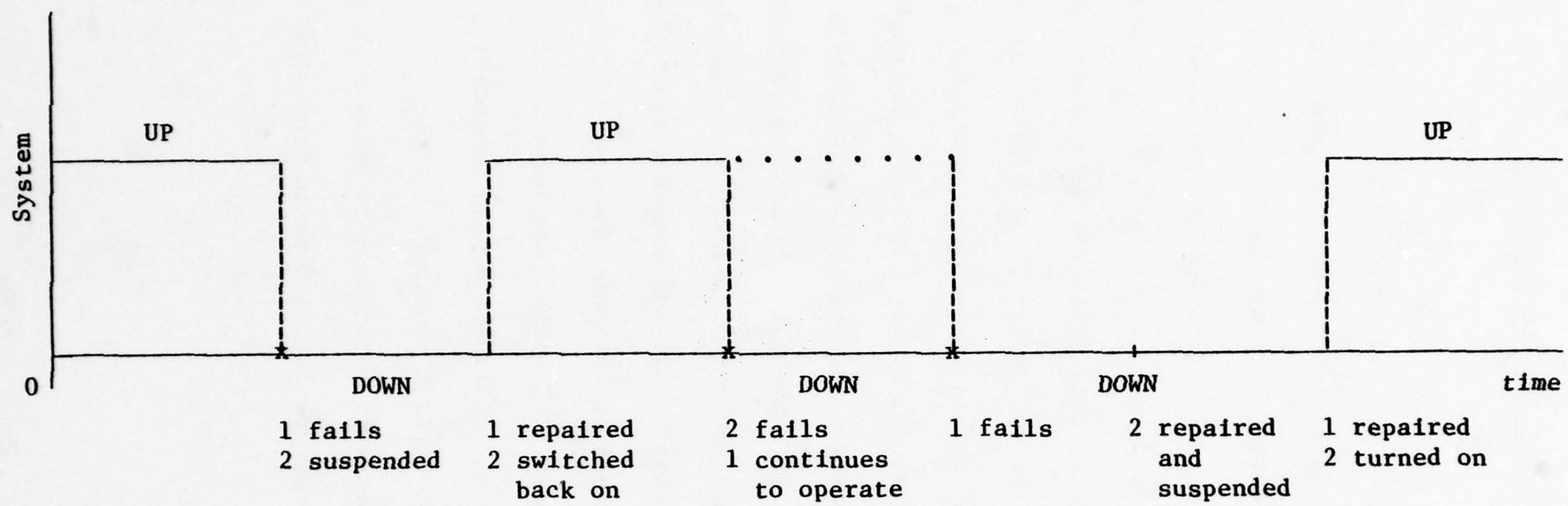

FIGURE 2.1

SAMPLE HISTORY OF TWO UNIT SERIES SYSTEM

$Z(t) = 0$ means both components are working at time t ;

$Z(t) = 1$ means component 1 alone is failed (and component 2 is suspended);

$Z(t) = 2$ means component 2 alone is failed;

$Z(t) = 3$ means both components are failed and undergoing repair.

It will sometimes be useful to split state 1 into 1' and 1'', where

$Z(t) = 1'$ means at the last state transition component 1 failed while 2 was working;

$Z(t) = 1''$ means at the last state transition, component 1 failed while 2 was under repair.

For definiteness we assume all components are new and begin working at time $t = 0$. The limiting results will, of course, be independent of this condition.

Let $p_j(t) = \Pr [Z(t) = j]$, $j = 0,1,2,3$ and $j = 1',1''$. Of course

$$p_1(t) = p_{1'}(t) + p_{1''}(t)$$

and $p_0(0) = 1$, while $p_j(0) = 0$ for $j \neq 0$. Let

$$\pi_j = \lim_{t\to\infty} p_j(t) \qquad j = 0,1,2,3,1',1'' \tag{2.1.1}$$

be the stationary probabilities when they exist.

Let $\lambda_i(t)$ be the failure rate for component i and $\theta_i(t)$ be the repair rate for component i $(i = 1,2)$ when they exist. We have the following state space diagrams for our process.

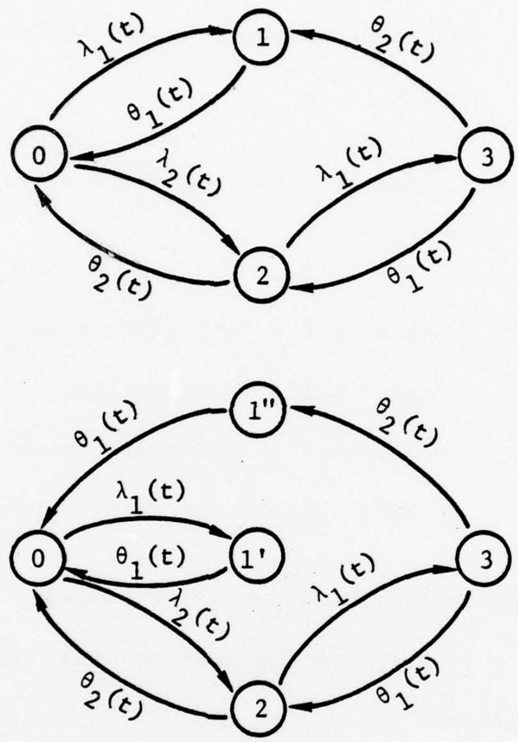

FIGURE 2.2

STATE SPACE DIAGRAMS FOR TWO UNIT SYSTEMS

<u>Remark</u>:

For the exponential case, all intensities are constant, so that the quantities on the arrows are well defined. For the general case, the quantities should be understood *conditionally* on being at a given state at time 0 (the time of the last state transition).

Notation:

We are primarily interested in asymptotic measures of system performance, when they exist, such as

$$A \overset{\text{def}}{=} \lim_{t\to\infty} p_0(t) = \pi_0 ,$$

the limiting system availability.

Let $U_1, U_2, \ldots, U_n, \ldots$ be successive system uptime intervals and $\tilde{N}(t)$ the number of system failures in $[0,t]$. The asymptotic average of system uptimes will be

$$\mu \overset{\text{def}}{=} \lim_{t\to\infty} \frac{\sum_{i=1}^{\tilde{N}(t)} U_i}{\tilde{N}(t)} . \tag{2.1.2}$$

Let $D_1, D_2, \ldots, D_n, \ldots$ be successive system downtime intervals. The asymptotic average of system downtimes will be

$$\nu \overset{\text{def}}{=} \lim_{t\to\infty} \frac{\sum_{i=1}^{\tilde{N}(t)} D_i}{\tilde{N}(t)} . \tag{2.1.3}$$

We are interested in comparing these measures under various distribution assumptions. By $A(F_1,G_1)$, $\mu(F_1,G_1)$, $\nu(F_1,G_1)$ we mean the asymptotic measures computed under the assumptions that F_1 and G_1 are general distributions and all other distributions are exponential. These measures will sometimes depend on the Laplace transforms

$$\bar{F}_i^*(s) = \int_0^\infty \bar{F}_i(t)e^{-st}dt = \frac{1 - f_i^*(s)}{s} \tag{2.1.4}$$

and

(2.1.5) $$\bar{G}_i^*(s) = \int_0^\infty \bar{G}_i(t)e^{-st}dt = \frac{1 - g_i^*(s)}{s}$$

where $\bar{F}_i = 1 - F_i$ and

$$f_i^*(s) = \int_0^\infty e^{-st}dF_i(t)$$

$(i = 1,2)$.

It will sometimes be convenient to let $\lambda_i = 1/\mu_i$ and $\theta_i = 1/\nu_i$ where μ_i (ν_i) is the mean life (mean repair time) of component i $(i = 1,2)$. In the exponential case these are of course the failure (repair) rates. Also let $\rho_i = \lambda_i/\theta_i$. By $A(\exp)$, $\mu(\exp)$, $\nu(\exp)$, we mean that corresponding measures are computed under the assumption that *all* distributions are exponential.

The shut-off rule considered in this chapter can be diagrammed as

$$\underset{1}{\circ} \rightarrow \underset{2}{\circ}$$

meaning 1 shuts off 2 but not vice versa. A similar notation is used with subscripts, as

$$A_{\cdot\cdot} \leq A_{\rightarrow} \leq A_{\leftrightarrow}$$

meaning that availability computed under component independence is less than or equal to availability under the rule that 1 shuts off 2, but not vice versa, etc.

Availability is greatest when each component can shut off the other. The inequalities are conjectured to hold in general although a proof for the general case is missing.

Let U_{ij} (D_{ij}) $j = 1,2, \ldots$ be successive up (down) times of component i $(i = 1,2)$. Let $U(t)$ $(D(t))$ be the cumulative system up (down) time in $[0,t]$. Let $\tilde{N}i(t)$ be the cumulative number of system failures caused by component i in $[0,t]$ $(i = 1,2)$ and let $\tilde{N}(t) = \tilde{N}1(t) + \tilde{N}2(t)$ be the cumulative number of system failures in $[0,t]$.

We define a few more related quantities later in the chapter, as the need arises.

2.2 General Failure and Repair Distributions

In this section we establish the existence of the stationary probabilities π_j $(j = 0,1,2,3,1',1'')$ under some assumptions on the distributions F_i, G_i $(i = 1,2)$. We also give some relationships among the π_j's, and obtain expressions for a few asymptotic measures of performance of the system in terms of these probabilities, such as the limiting availability, the limiting system failure rate, the asymptotic relative importance of component 2 to component 1, the limiting average of system up (down) times. Finally we compare these respective measures under the three shut-off rules.

Assumptions on the distributions F_i, G_i $(i = 1,2)$ are needed to ensure existence of regeneration points. A regeneration point occurs if both components are under repair and component 2 is repaired before component 1. At the moment 1 is repaired, both are then like new and this is a regeneration point. See Figure 2.3.

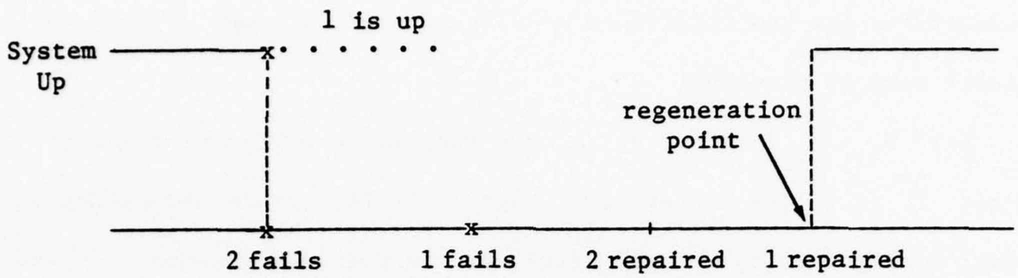

FIGURE 2.3

OCCURRENCE OF REGENERATION POINT

Let $T_1, T_2, \ldots$ be successive times between regeneration points.

Theorem 2.2.1

Assume the regeneration time T_1 satisfies $ET_1 < \infty$ and T_1 has an absolutely continuous component. Then π_j $(j = 0,1,2,3,1',1'')$ exist. (For example, if $0 < \mu_i , \nu_i < \infty$ $(i = 1,2)$ and F_1 is absolutely continuous and strictly increasing on $[0,\infty)$, then the assumptions hold.)

Proof:

See Ross (1970), Theorem 5.8, p. 95. ■

In all that follows we assume the conditions of the theorem hold and that $0 < \mu_i , \nu_i < \infty$ $(i = 1,2)$.

We next prove that the long run average system uptime converges to π_0 , the stationary probability that the system is up.

Theorem 2.2.2

$$\lim_{t\to\infty} \frac{U(t)}{t} = \pi_0 \quad \text{almost surely (a.s.)} . \tag{2.2.1}$$

Proof:

Let $K(t)$ be the number of complete regeneration cycles in $[0,t]$. Let U_r, $r = 1,2, \ldots$ be the cumulative system uptime in the r^{th} regeneration cycle. Clearly $\{U_r\}_{r=1}^{\infty}$ is a renewal process. We note that

$$\sum_{r=1}^{K(t)} U_r \leq U(t) \leq \sum_{r=1}^{K(t)+1} U_r .$$

The left-hand inequality results from the fact that $U(t)$ may include a partial uptime of the $K(t)$ + 1st regeneration cycle. The right-hand inequality results from the fact that the final portion of the uptime by time t may actually be smaller than $U_{K(t)+1}$.

From the right-hand inequality,

$$\underline{\lim}_{t\to\infty} \frac{U(t)}{t} \geq \lim_{t\to\infty} \frac{K(t)}{t} \cdot \left[\sum_{r=1}^{K(t)} U_r\right] / K(t)$$

$$= EU_1/ET_1 \qquad \text{a.s.}$$

by the strong law of large numbers and Theorem 6.3.6 of Barlow and Proschan (1975), p. 167. Similarly,

$$\overline{\lim}_{t\to\infty} \frac{U(t)}{t} \leq EU_1/ET_1 \qquad \text{a.s.}$$

so that

$$\lim_{t\to\infty} \frac{U(t)}{t} = EU_1/ET_1 \qquad \text{a.s.}$$

By Theorem 5.8 of Ross (1970),

$$\pi_0 = \frac{E[\text{amount of time spent in state 0 during one cycle}]}{E[\text{length of one cycle}]}$$

$$= EU_1/ET_1$$

so that the theorem is proved.■

Corollary 2.2.3:

$$\lim_{t\to\infty} \frac{EU(t)}{t} = \pi_0 \; .$$

Proof:

Since $\frac{U(t)}{t} \leq 1$ and $\lim \frac{U(t)}{t} = \pi_0$ by Theorem 2.2.2, the result follows from the Lebesgue dominated convergence theorem.■

In order to find the asymptotic system failure rate, $\lim_{t\to\infty} \frac{\tilde{N}(t)}{t}$, we need first to find the asymptotic rates at which components 1 and 2 are causing system failures. Let $\widetilde{N2}(t)$ be the cumulative number of system failures due to component 2 in $[0,t]$.

Theorem 2.2.4:

$$\lim_{t\to\infty} \frac{\widetilde{N2}(t)}{t} = \lambda_2\pi_0 \quad \text{a.s.} \tag{2.2.2}$$

$$\lim_{t\to\infty} \frac{\widetilde{N2}(t)}{U(t)} = \lambda_2 \quad \text{a.s.} \tag{2.2.3}$$

Proof:

Let U_{2r} be successive lifetimes (excluding intervals of suspended animation) of component 2, so that

$$\frac{1}{N2(t)} \sum_{r=1}^{\widetilde{N2}(t)} U_{2r} \leq \frac{U(t)}{\widetilde{N2}(t)} \leq \frac{1}{\widetilde{N2}(t)} \sum_{r=1}^{\widetilde{N2}(t)+1} U_{2r} .$$

Note that the system uptimes coincide with component 2 uptimes. Since $\widetilde{N2}(t) \to \infty$ a.s. and $\{U_{2r}\}_{r=1}^{\infty}$ is a renewal process

$$\lim_{t\to\infty} U(t)/\widetilde{N2}(t) = \mu_2 . \qquad \text{a.s.}$$

Now

$$\lim_{t\to\infty} \frac{\widetilde{N2}(t)}{t} = \lim_{t\to\infty} \frac{\widetilde{N2}(t)}{U(t)} \cdot \frac{U(t)}{t} = \lambda_2 \pi_0 \qquad \text{a.s.}$$

by the strong law of large numbers and Theorem 2.2.2. ∎

When component 1 fails and causes system failure, 2 is suspended until 1 is repaired. Let $S'(t)$ be the cumulative sum of such suspension times in $[0,t]$. Also, if both components are under repair and 2 is repaired first, 2 is again suspended until 1 is repaired. Let the cumulative sum of these second kind of suspension times in $[0,t]$ be $S''(t)$. Recall that state 1' occurs when 1 fails causing system failure.

Lemma 2.2.5:

$$\lim_{t\to\infty} \frac{S'(t)}{t} = \pi_{1'} . \qquad \text{a.s.} \tag{2.2.4}$$

Proof:

Let S'_r , $r = 1,2, \ldots$ be the cumulative suspension time of component 2 of the first kind during the r^{th} regeneration cycle.

Recall that $K(t)$ is the number of complete regeneration cycles in $[0,t]$, so that

$$\frac{1}{t}\sum_{r=1}^{K(t)} S'_r \le \frac{S'(t)}{t} \le \frac{1}{t}\sum_{r=1}^{K(t)+1} S'_r .$$

By the strong law of large numbers and Theorem 6.3.6 of Barlow and Proschan (1975)

$$\lim_{t\to\infty} \frac{S'(t)}{t} = \frac{E[S'_r]}{ET_1} \qquad \text{a.s.}$$

Again by Theorem 5.8 of Ross (1970)

$$\lim_{t\to\infty} \frac{S'(t)}{t} = \pi_1,$$

so that the theorem is proved. ∎

<u>Theorem 2.2.6</u>:

$$\lim_{t\to\infty} \frac{\widetilde{N1}(t)}{t} = \theta_1\pi_1, \qquad \text{a.s.} \tag{2.2.5}$$

<u>Proof</u>:

Let $D_{11}, D_{12}, \ldots$ be consecutive repair times of component 1 after 1 has suspended 2, i.e., there are first kind of suspension periods. Note that $\{D_{1r}\}_{r=1}^{\infty}$ is a renewal process and

$$\sum_{r=1}^{\widetilde{N1}(t)-1} D_{1r} \le S'(t) \le \sum_{r=1}^{\widetilde{N1}(t)} D_{1r}$$

since a suspension period for component 2 of the first kind is a complete repair of component 1. By the previous lemma and the strong law of large numbers,

$$\lim_{t\to\infty} \frac{\widetilde{N1}(t)}{t} = \lim_{t\to\infty} \frac{\widetilde{N1}(t)}{S'(t)} \cdot \frac{S'(t)}{t} = \theta_1 \pi_{1'}, \quad \text{a.s.} \blacksquare$$

<u>Corollary 2.2.7</u>:

The asymptotic system failure rate is

$$\lim_{t\to\infty} \frac{\tilde{N}(t)}{t} = \lambda_1 \pi_0 \frac{\theta_1 \pi_{1'}}{\lambda_1 \pi_0} + \lambda_2 \pi_0 \quad \text{a.s.} \tag{2.2.6}$$

<u>Remark</u>:

It is interesting to note that when F_1 is exponential (see next section) $\dfrac{\theta_1 \pi_{1'}}{\lambda_1 \pi_0} = 1$ so that in this case the asymptotic system failure rate is of similar form for all three possible shut-off rules for two components in series. (Of course π_0 is different in each case.)

<u>Proof</u>:

By Theorems 2.2.4 and 2.2.6

$$\lim_{t\to\infty} \frac{\tilde{N}(t)}{t} = \lim_{t\to\infty} \left(\frac{\widetilde{N1}(t)}{t} + \frac{\widetilde{N2}(t)}{t} \right)$$

$$= \theta_1 \pi_{1'} + \lambda_2 \pi_0 \quad \text{a.s.} \blacksquare$$

<u>Corollary 2.2.8</u>:

The asymptotic relative importance of component 2 to component 1 is

(2.2.7)
$$\lim_{t\to\infty} \frac{\widetilde{N}2(t)}{\widetilde{N}1(t)} = \frac{\lambda_2\pi_0}{\theta_1\pi_{1'}} \quad \text{a.s.}$$

Remark:

If F_1 is exponential, then it turns out that $\dfrac{\theta_1\pi_{1'}}{\lambda_1\pi_0} = 1$ so that in this case

(2.2.8)
$$\lim_{t\to\infty} \frac{\widetilde{N}2(t)}{\widetilde{N}1(t)} = \frac{\lambda_2}{\lambda_1} \quad \text{a.s.}$$

In Part II, Theorem 2.2, we show that if F_1 is NBUE (NWUE), G_1 is general, and F_2 , G_2 are exponential then

$$\frac{\theta_1\pi_{1'}}{\lambda_1\pi_0} \leq (\geq)\ 1$$

so that

$$\lim_{t\to\infty} \frac{\widetilde{N}2(t)}{\widetilde{N}1(t)} \leq (\geq)\ \frac{\lambda_2}{\lambda_1} \quad \text{a.s.}$$

Hence, if F_1 is NBUE (NWUE) and $\lambda_1 \geq \lambda_2$ $(\lambda_1 \leq \lambda_2)$ then component 1 (2) is more important.

We will need the following lemma.

Lemma 2.2.9:

(2.2.9)
$$\pi_2 + \pi_3 = \frac{\lambda_2}{\theta_2}\pi_0 \ .$$

Proof:

$\pi_2 + \pi_3$ is the limiting probability that component 2 is under repair. By an argument similar to that of Theorem 2.2.2, we have

$$\pi_2 + \pi_3 = \lim_{t\to\infty} \frac{D_2(t)}{t} \quad \text{a.s.}$$

where $D_2(t)$ is the cumulative downtime of component 2 in $[0,t]$. Also

$$\frac{D_2(t)}{t} \doteq \frac{1}{t} \sum_{r=1}^{\tilde{N}2(t)} D_{2r} = \frac{\tilde{N}2(t)}{t} \frac{1}{\tilde{N}2(t)} \sum_{r=1}^{\tilde{N}2(t)} D_{2r}$$

where $\{D_{2r}\}_{r=1}^{\infty}$ is the renewal process of repair times for component 2. By Theorem 2.2.4 and the strong law of large numbers

$$\lim_{t\to\infty} \frac{D_2(t)}{t} = \lambda_2\pi_0/\theta_2 \quad \text{a.s.}$$

Theorem 2.2.10:

The limiting average of system uptimes (downtimes) is μ (ν) where

(2.2.10) $$\mu = \frac{1}{\lambda_1 \dfrac{\theta_1\pi_1'}{\lambda_1\pi_0} + \lambda_2}$$

and

(2.2.11) $$\nu = \frac{1 - \pi_0}{\left(\lambda_1 \dfrac{\theta_1\pi_1'}{\lambda_1\pi_0} + \lambda_2\right)\pi_0} .$$

Remark:

Using Theorem 2.2.10, we can show

$$\frac{\mu}{\mu + \nu} = \pi_0$$

as we would expect.

Proof:

It is easy to see that

$$\mu = \lim_{t\to\infty} \frac{U(t)}{\tilde{N}(t)} = \lim_{t\to\infty} \frac{U(t)}{t} \frac{t}{\tilde{N}(t)}$$

$$= \frac{1}{\lambda_1 \frac{\theta_1 \pi_{1'}}{\lambda_1 \pi_0} + \lambda_2} \quad \text{a.s.}$$

by Theorem 2.2.2 and Corollary 2.2.7. Also

$$\nu = \lim_{t\to\infty} \frac{D(t)}{\tilde{N}(t)} = \lim_{t\to\infty} \frac{t - U(t)}{\tilde{N}(t)}$$

$$= \frac{1 - \pi_0}{\left(\lambda_1 \frac{\theta_1 \pi_{1'}}{\lambda_1 \pi_0} + \lambda_2\right)\pi_0} \quad \text{a.s.}$$

by Theorem 2.2.2 and Corollary 2.2.7. Using Lemma 2.2.9 we see that

$$1 - \pi_0 = \pi_1 + (\pi_2 + \pi_3) = \pi_1 + \rho_2 \pi_0$$

$$= \left(\rho_1 \frac{\theta_1 \pi_1}{\lambda_1 \pi_0} + \rho_2\right)\pi_0 \ . \blacksquare$$

Corollary 2.2.11:

Using subscripts to denote shut-off rules we have

$$\mu_{\rightarrow} > \mu_{\leftrightarrow} = \mu_{\cdot\cdot} = \frac{1}{\lambda_1 + \lambda_2}$$

if and only if $\frac{\theta_1 \pi_{1'}}{\lambda_1 \pi_0} < 1$.

Remark:

We will show that the ratio is equal to 1 when F_1 is exponential and can be either greater or less than 1 depending on F_1 .

Proof:

This is obvious from examining (2.2.10).■

Next we compare limiting availability for several shut-off rules. We will need the following lemma.

Lemma 2.2.12:

$$\pi_0 + \pi_2 = \frac{\theta_1}{\lambda_1 + \theta_1} \tag{2.2.12}$$

$$\pi_1 + \pi_3 = \frac{\lambda_1}{\lambda_1 + \theta_1} \ . \tag{2.2.13}$$

Proof:

$\pi_0 + \pi_2$ is the limiting probability that component 1 is functioning. Hence, by the Key Renewal Theorem for alternating renewal processes we get (2.2.12).

$\pi_1 + \pi_3$ is the limiting probability that component 1 is under repair, and since $(\pi_0 + \pi_2) + (\pi_1 + \pi_3) = 1$, (2.2.13) follows.■

Corollory 2.2.13:

$$A_{\cdot\cdot} \leq A_{\rightarrow} \ . \tag{2.2.14}$$

Proof:

By Theorem 2.2.10 and the remark following, (2.2.14) holds if and only if

$$\frac{\nu_{\cdot\cdot}}{\mu_{\cdot\cdot}} \geq \frac{\nu_{+}}{\mu_{+}}$$

or

$$\rho_1 + \rho_2 + \rho_1\rho_2 \geq \rho_1 \frac{\theta_1\pi_1}{\lambda_1\pi_0} + \rho_2$$

or

$$1 + \rho_2 \geq \frac{\theta_1\pi_1}{\lambda_1\pi_0} ,$$

$$\lambda_1\pi_0(1 + \rho_2) \geq \theta_1\pi_1 .$$

By Lemma 2.2.9,

$$\lambda_1\pi_0(1 + \rho_2) = \lambda_1(\pi_0 + \pi_2 + \pi_3) = \lambda_1(1 - \pi_1)$$

so the inequality is equivalent to

$$\lambda_1(1 - \pi_1) \geq \theta_1\pi_1$$

or

$$\pi_1 \leq \frac{\lambda_1}{\lambda_1 + \theta_1}$$

which holds by Lemma 2.2.12. ■

<u>Corollary 2.2.14</u>:

(2.2.15) $$A_{\rightarrow} \leq A_{\leftrightarrow}$$

if and only if

$$\frac{\theta_1 \pi_1}{\lambda_1 \pi_0} \geq 1 \ .$$

<u>Remark</u>:

We conjecture that (2.2.15) always holds. This makes sense intuitively, since by shutting off both components we save more component life. We show later that $\theta_1\pi_1/\lambda_1\pi_0 \geq 1$ when F_1 is exponential (see Corollary 2.4.2) or when F_2 and G_2 are exponential and F_1 or G_1 is NWUE (see Part II). We were unable to prove the conjecture for general F_i , G_i $(i = 1,2)$.

<u>Proof</u>:

By Theorem 2.2.10 and the remark following, (2.2.15) holds if and only if

$$\frac{\nu_{\rightarrow}}{\mu_{\rightarrow}} \geq \frac{\nu_{\leftrightarrow}}{\mu_{\leftrightarrow}}$$

or

$$\rho_1 \frac{\theta_1 \pi_1}{\lambda_1 \pi_0} + \rho_2 \geq \rho_1 + \rho_2 \ ,$$

or

$$\frac{\theta_1 \pi_1}{\lambda_1 \pi_0} \geq 1 \ . \blacksquare$$

2.3 Exponential Failure and Repair Distributions

In this section we assume that F_i , G_i , $i = 1,2$ are all exponential. We refer to this case later as the "all-exponential case." Here it is easy to obtain the stationary probabilities and measures of system performance.

The following results should be compared with the results for the more general distribution assumptions.

Theorem 2.3.1:

The stationary state probabilities are given by

(2.3.1)
$$\pi_0 = \frac{\theta_1}{\lambda_1 + \theta_1}\left(\frac{\theta_2(\lambda_1 + \theta_1 + \theta_2)}{\theta_2(\lambda_1 + \theta_1 + \theta_2) + \lambda_2(\theta_1 + \theta_2)}\right).$$

(2.3.2)
$$\begin{aligned}\pi_1 &= \frac{\lambda_1}{\lambda_1 + \theta_1}\left(\frac{\theta_2(\lambda_1 + \lambda_2 + \theta_1 + \theta_2)}{\theta_2(\lambda_1 + \lambda_2 + \theta_1 + \theta_2) + \lambda_2\theta_1}\right)\\ &= \frac{\lambda_1}{\theta_1}\left(\frac{\lambda_1 + \lambda_2 + \theta_1 + \theta_2}{\lambda_1 + \theta_1 + \theta_2}\right)\pi_0 .\end{aligned}$$

(2.3.3)
$$\begin{aligned}\pi_2 &= \frac{\theta_1}{\lambda_1 + \theta_1}\left(\frac{\lambda_2(\theta_1 + \theta_2)}{\lambda_2(\theta_1 + \theta_2) + \theta_2(\lambda_1 + \theta_1 + \theta_2)}\right)\\ &= \frac{\lambda_2}{\theta_2}\left(\frac{\theta_1 + \theta_2}{\lambda_1 + \theta_1 + \theta_2}\right)\pi_0 .\end{aligned}$$

(2.3.4)
$$\begin{aligned}\pi_3 &= \frac{\lambda_1}{\lambda_1 + \theta_1}\left(\frac{\lambda_2\theta_1}{\theta_2(\lambda_1 + \lambda_2 + \theta_1 + \theta_2) + \lambda_2\theta_1}\right)\\ &= \frac{\lambda_2}{\theta_2}\left(\frac{\lambda_1}{\lambda_1 + \theta_1 + \theta_2}\right)\pi_0 .\end{aligned}$$

Also,

$$(2.3.5)\qquad \begin{aligned}\pi_{1'} &= \frac{\lambda_1}{\lambda_1+\theta_1}\left(\frac{\theta_2(\lambda_1+\theta_1+\theta_2)}{\theta_2(\lambda_1+\theta_1+\theta_2)+\lambda_2(\theta_1+\theta_2)}\right)\\ &= \frac{\lambda_1}{\theta_1}\pi_0 \ .\end{aligned}$$

$$(2.3.6)\qquad \begin{aligned}\pi_{1''} &= \frac{\lambda_1}{\lambda_1+\theta_1}\left(\frac{\lambda_2\theta_2}{\theta_2(\lambda_1+\theta_1+\theta_2)+\lambda_2(\theta_1+\theta_2)}\right)\\ &= \frac{\lambda_1}{\theta_1}\left(\frac{\lambda_2}{\lambda_1+\theta_1+\theta_2}\right)\pi_0 \ .\end{aligned}$$

Proof:

From Lemma 2.2.12,

$$(2.3.7)\qquad \pi_0+\pi_2 = \frac{\theta_1}{\lambda_1+\theta_1} \ .$$

From Lemma 2.2.9,

$$(2.3.8)\qquad \pi_2+\pi_3 = \rho_2\pi_0 \ .$$

Since the stationary flow into and out of state 1 must be equal,

$$(2.3.9)\qquad \theta_1\pi_1 = \lambda_1\pi_0+\theta_2\pi_3$$

and, of course

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 . \tag{2.3.10}$$

Solving these four equations yields the expressions for π_i for $i = 0,1,2,3$ stated in the theorem.

To obtain expressions for $\pi_{1'}$, $\pi_{1''}$, note that

$$\theta_1\pi_{1'} = \lambda_1\pi_0 \tag{2.3.11}$$

$$\theta_1\pi_{1''} = \theta_2\pi_3 , \tag{2.3.12}$$

since these are the flow rates into and out of states 1', 1'', respectively.■

<u>Corollary 2.3.2</u>:

In the all-exponential case

$$\text{(i)} \qquad \mu = \frac{1}{\lambda_1 + \lambda_2} . \tag{2.3.13}$$

$$\text{(ii)} \qquad \nu = \frac{\rho_1 \dfrac{\lambda_1 + \lambda_2 + \theta_1 + \theta_2}{\lambda_1 + \theta_1 + \theta_2} + \rho_2}{\lambda_1 + \lambda_2} . \tag{2.3.14}$$

<u>Proof</u>:

(i) follows from (2.2.10) and (2.2.5). (ii) follows from (2.2.11), (2.3.1) and (2.3.5).■

We next compare $A_{\rightarrow}$, $\mu_{\rightarrow}$, $\nu_{\rightarrow}$ in the all-exponential case with the respective measures of system performance for the other shut-off rules.

These measures are:

Remark 2.3.3:

(i) $\mu_{\leftrightarrow} = \mu_{\cdot\cdot} = \dfrac{1}{\lambda_1 + \lambda_2}$.

(ii) $\nu_{\leftrightarrow} = \dfrac{\rho_1 + \rho_2}{\lambda_1 + \lambda_2}$,

$$\nu_{\cdot\cdot} = \frac{\rho_1 + \rho_2 + \rho_1\rho_2}{\lambda_1 + \lambda_2},$$

$$\nu_{\cdot\cdot} \geq \nu_{\leftrightarrow}.$$

(iii) $A_{\leftrightarrow} = \dfrac{1}{1 + \rho_1 + \rho_2}$,

$$A_{\cdot\cdot} = \frac{1}{(1 + \rho_1)(1 + \rho_2)},$$

$$A_{\cdot\cdot} \leq A_{\leftrightarrow}.$$

(iv) $$A_{\rightarrow} = \frac{1}{(1 + \rho_1)\left[1 + \rho_2 \dfrac{\theta_1 + \theta_2}{\lambda_1 + \theta_1 + \theta_2}\right]}.$$

Corollary 2.3.4:

In the all-exponential case

(i) $A_{\cdot\cdot} \leq A_{\rightarrow} \leq A_{\leftrightarrow}$.

(ii) $\mu_{\cdot\cdot} = \mu_{\rightarrow} = \mu_{\leftrightarrow}$.

(iii) $\nu_{\cdot\cdot} \geq \nu_{\rightarrow} \geq \nu_{\leftrightarrow}$.

Proof:

(i) follows by comparing (iii) and (iv) of Remark 2.3.3.
(ii) follows by comparing (2.3.13) and (i) of Remark 2.3.3.
(iii) follows by comparing (2.3.14) and (ii) of Remark 2.3.3.■

Corollary 2.3.5:

The limiting results for F_1 , G_1 , G_2 exponential and F_2 general are the same as for the all-exponential case.

Proof:

The proof of Theorem 2.3.1 was based on the six equations (2.3.7) - (2.3.12). Three of these, namely (2.3.7), (2.3.8), (2.3.10), hold for any four distributions. The other three involve only the constants λ_1 , θ_1 , θ_2 , i.e., the exponential rates. They do not involve λ_2 explicitly, hence they will hold even if F_2 is not exponential.■

2.4 F_1 Exponential

In this section we assume that F_1 is exponential and all other distributions are general. For this case, certain asymptotic measures of system performance are similar for all three shut-off rules, i.e., for all three rules

$$\lim_{t\to\infty} \frac{\tilde{N}(t)}{t} = (\lambda_1 + \lambda_2)\pi_0 \qquad \text{a.s.}$$

$$\lim_{t\to\infty} \frac{\widetilde{N2}(t)}{\widetilde{N1}(t)} = \frac{\lambda_2}{\lambda_1} \qquad \text{a.s.}$$

and

$$\mu = 1/(\lambda_1 + \lambda_2)$$

although π_0 differs for different shut-off rules. The key result is contained in

Theorem 2.4.1:

If F_1 is exponential

$$\theta_1\pi_{1'} = \lambda_1\pi_0$$

so that

$$\lim_{t\to\infty} \frac{\widetilde{N1}(t)}{t} = \lambda_1\pi_0 \qquad \text{a.s.}$$

Proof:

Our two component series system stochastic process can be realized in terms of a convenient Poisson process $\{N^0(t) ; t \geq 0\}$ with intensity λ_1 . Start the N^0 process and component 2 at $t = 0$. The first event from the N^0 process corresponds to a component 1 failure. Events from N^0 during repair of component 1 are ignored. The first event from N^0 after repair of 1 corresponds to another component 1 failure, etc. Figure 2.4 illustrates the idea.

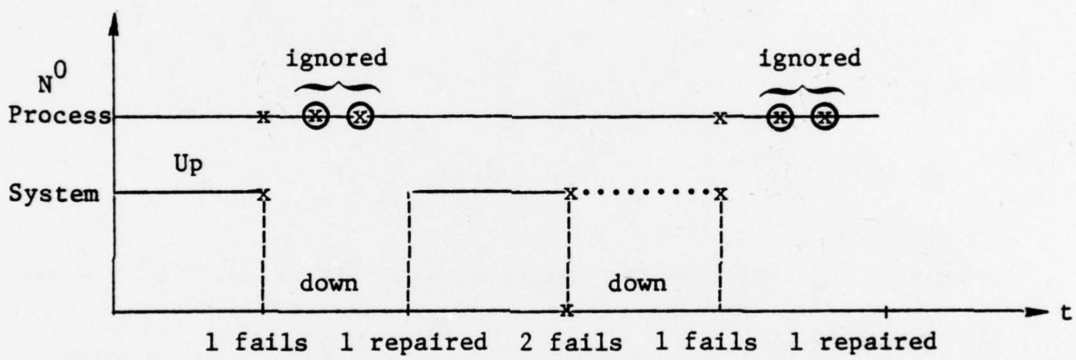

FIGURE 2.4

Let $N_1^0(t)$ be the cumulative number of events from the N^0 process in $[0,t]$ which shut off the system, i.e., suspend component 2. Then $N_1^0(t)$ and $\tilde{N}1(t)$ are stochastically equal. Let $N_2^0(t) = N^0(t) - N_1^0(t)$ and $D(t)$ the cumulative system downtime in $[0,t]$. Then $D(t) = \sum_{r=1}^{\tilde{N}(t)} D_r$ and $N_2^0(t) = \sum_{r=1}^{\tilde{N}(t)} N_2^0(D_r)$ where $\tilde{N}(t)$ is the number of system failures in $[0,t]$, D_r $r = 1,2, \ldots$ are the successive downtimes of the system in $[0,t]$ ($D_{\tilde{N}(t)}$ is truncated)

and by the ambiguous notation $N_2^0(D_r)$ we mean the number of events from N_2^0 during the r^{th} downtime of the system. Then

$$E\left[N_2^0(D_r) \mid D_j , j \leq r\right] = \lambda_1 D_r$$

by the strong Markov property of the Poisson process, since D_r is a stopping time for $N_2^0(D_r)$ for each r. [Cf. Çinlar (1975)].

Hence

$$E\left[N_2^0(t) \mid D(t)\right] = \sum_{r=1}^{\tilde{N}(t)} E\left[N_2^0(D_r) \mid D(t)\right]$$

$$= \sum_{r=1}^{\tilde{N}(t)} E\left[N_2^0(D_r) \mid D_j , j \leq r\right] = \sum_{r=1}^{\tilde{N}(t)} \lambda_1 D_r = \lambda_1 D(t) ,$$

so that

$$EN_2^0(t) = \lambda_1 ED(t)$$

and

$$EN_1^0(t) = EN^0(t) - EN_1^0(t) = \lambda_1 t - \lambda_1 ED(t)$$

$$= \lambda_1 EU(t) .$$

It follows that

$$\lim_{t\to\infty} \frac{EN_1^0(t)}{t} = \lambda_1 \pi_0 = \lim_{t\to\infty} \frac{E\tilde{N}1(t)}{t} .$$

By Theorem 2.2.6

$$\lim_{t\to\infty} \frac{\widetilde{N1}(t)}{t} = \theta_1\pi_{1'}$$

so we must have $\theta_1\pi_{1'} = \lambda_1\pi_0$ which completes the proof.■

Corollary 2.4.2:

If F_1 is exponential

(i) $\lim_{t\to\infty} \frac{\tilde{N}(t)}{t} = (\lambda_1 + \lambda_2)\pi_0$ a.s.

(ii) $\lim_{t\to\infty} \frac{\widetilde{N2}(t)}{\widetilde{N1}(t)} = \frac{\lambda_2}{\lambda_1}$ a.s.

(iii) $\mu = \frac{1}{\lambda_1 + \lambda_2}$

(iv) $\nu_{\cdot\cdot} \geq \nu_{\rightarrow} \geq \nu_{\leftrightarrow}$

(v) $A_{\rightarrow} \leq A_{\leftrightarrow}$.

Proof:

(i) follows from Corollary 2.2.7. (ii) follows from Corollary 2.2.8. (iii) follows from Corollary 2.2.11. (iv) follows from Theorem 2.2.10 and the inequality $\theta_1\pi_1 = \theta_1\pi_{1'} + \theta_1\pi_{1''} > \theta_1\pi_{1'} = \lambda_1\pi_0$.
(v) follows from (iii) and (iv).■

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